Running Coupling Effects in Small-x QCD

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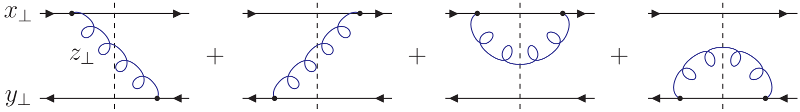
Outline

- Many approaches to small-x QCD

- Towards the next to leading order BK equation
- Running coupling effects on the Pomeron intercept $\rightsquigarrow$ Sensitivity to infrared
- The saturation momentum and geometric scaling
- Running coupling vs. Pomeron loop effects

TOWARDS THE NLO BK EQUATION

Leading order



- Probability for soft gluon emission in the dipole wavefunction

$$\mathrm{d}P = \frac{\bar{\alpha}}{2\pi} \underbrace{\frac{(\boldsymbol{x}-\boldsymbol{y})^2}{(\boldsymbol{x}-\boldsymbol{z})^2(\boldsymbol{z}-\boldsymbol{y})^2}}_{\mathcal{M}_{\boldsymbol{xyz}}} \mathrm{d}^2 z \mathrm{d}Y$$

- Soft gluon $\rightarrow$ Quark-Antiquark pair

  Either daughter dipole can scatter off target hadron

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}}{2\pi} \int \mathrm{d}^2 \boldsymbol{z} \mathcal{M}_{\boldsymbol{xyz}} \left(S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}}\right)$$

Argument of coupling?

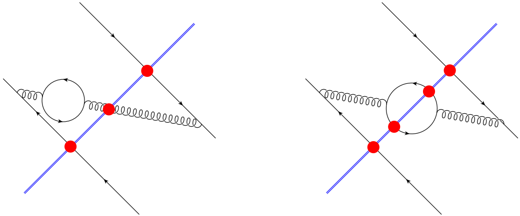
- Do not know scale in argument of coupling constant
  Non-local (in transverse space) evolution in contrast to DGLAP
- Expand running coupling to see what we need

$$\alpha(Q^2) = \alpha_\mu - \alpha_\mu^2 \, \beta \, \ln \frac{Q^2}{\mu^2} + \alpha_\mu^3 \, \beta^2 \, \ln^2 \frac{Q^2}{\mu^2} - \dots$$

- One quark-loop $\bar{\alpha}_\mu \times (\alpha_\mu N_f) \times \Delta Y$
  Two quark-loops $\bar{\alpha}_\mu \times (\alpha_\mu N_f)^2 \times \Delta Y$
  Sum $(\alpha_\mu N_f)^k$ for all k, then let $-2N_f \to 11N_c - 2N_f = 12\pi\beta$
- Recover scale in coupling argument

Quark loop

- Two classes of diagrams to order $\bar{\alpha}\alpha N_f$



- First type diagrams: typical running coupling correction

  Two contiguous dipoles $(\boldsymbol{x}, \boldsymbol{z})$ and $(\boldsymbol{z}, \boldsymbol{y})$

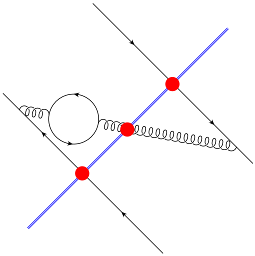
  Expect just a kernel modification to LO equation

- Second type diagrams: different wavefunction component $\leadsto$

  NLO equation: more complicate structure (plus double integration)

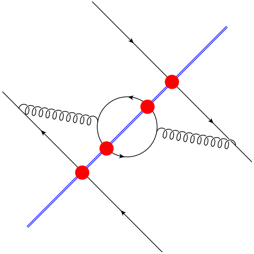
- Running coupling: two contiguous dipoles

Simple diagrams: running coupling



- Loop integration over $\boldsymbol{k}^2$: UV divergent
  Dimensional regularization $1/\epsilon \to \ln \mu^2$
- Integrate all longitudinal momenta of loop quark and antiquark

New channel diagrams

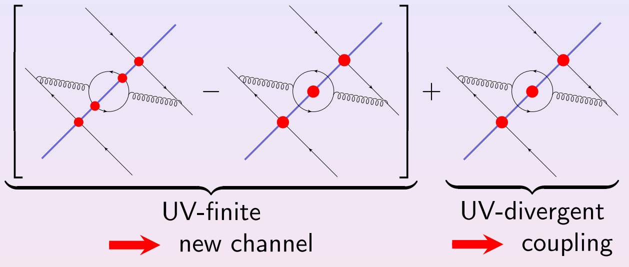


- When pair shrinks to a point $\rightsquigarrow$

  Size $\longrightarrow$ 0, loop momentum $\longrightarrow \infty$: UV divergent

  Contributes to running of coupling

Isolate running



- Add and subtract ∞ to isolate running
- Choose “point” as linear combination of q and $\bar{q}$ positions
- Not unique way (Balitsky vs Kovchegov-Weigert)
- Full NLO equation: Unique, but not closed
- Running coupling part: Closed equation, but not unique

NLO equation

- Next to leading order equation (Balitsky "scheme"):

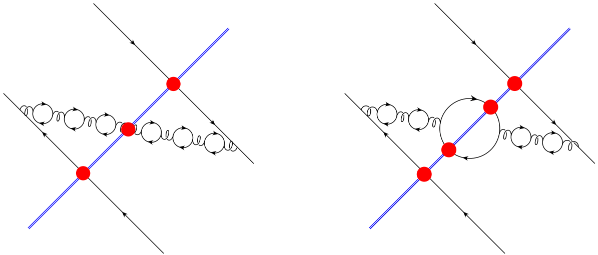
$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}_\mu}{2\pi} \int \mathrm{d}^2\boldsymbol{z} \mathcal{M}_{\boldsymbol{xyz}} \left[1 + \frac{\alpha_\mu N_f}{6\pi} \ln \frac{\mathrm{e}^{-5/3}}{(\boldsymbol{x}-\boldsymbol{y})^2 \mu^2} + \ldots \right] (S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})$$
$$+ \frac{\bar{\alpha}_\mu \alpha_\mu N_f}{N_c^2} \int \mathrm{d}^2\boldsymbol{z}_1 \mathrm{d}^2\boldsymbol{z}_2 \; [\text{new state}]$$

- Main difference in Kovchegov-Weigert "scheme" amounts to

$$\ln \frac{1}{(\boldsymbol{x}-\boldsymbol{y})^2 \mu^2} \to \ln \frac{R^2(\boldsymbol{r}_1, \boldsymbol{r}_2)}{\boldsymbol{r}_1^2 \boldsymbol{r}_2^2 \mu^2}$$

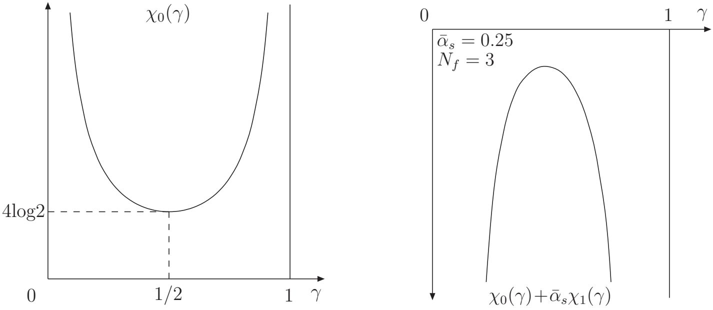
$\boldsymbol{r}_1$, $\boldsymbol{r}_2$: daughter dipole sizes

Bubbles



- Resum bubbles (contained in higher N^{n}LO corrections)
- How many resummations we need to do (still no P. loops)?
  - BFKL equation $\rightarrow$ Resum $(\bar{\alpha}Y)^n$
  - Non-linear terms $\leftrightarrow$ Resum target high density effects
  - Bubbles to get running coupling
  - Bad collinear behavior of NLO kernel $\rightarrow$ Pole resummation $\gamma(\omega = 1) = 0$

Characteristic function



- Act on $(\boldsymbol{r}^2)^{1-\gamma}$ (not an eigenfunction)
- More than obvious instability (even more complicated)

Argument of coupling

- Balitsky: Scale in coupling argument set by parent dipole size

  Running coupling equation:

$$\frac{\mathrm{d}S_{\boldsymbol{xy}}}{\mathrm{d}Y} = \frac{\bar{\alpha}(\boldsymbol{r}^2)}{2\pi} \int_{\boldsymbol{z}} \left\{ \mathcal{M}_{\boldsymbol{xyz}} + \frac{1}{\boldsymbol{r}_1^2} \left[\frac{\alpha(\boldsymbol{r}_1^2)}{\alpha(\boldsymbol{r}_2^2)} - 1 \right] + 1 \leftrightarrow 2 \right\} (S_{\boldsymbol{xz}} S_{\boldsymbol{zy}} - S_{\boldsymbol{xy}})$$

- Kovchegov-Weigert: Triumvirate of running couplings

$$\frac{\bar{\alpha}(\boldsymbol{r}_1^2)\bar{\alpha}(\boldsymbol{r}_2^2)}{\bar{\alpha}(R^2)}$$

IR cutoff

- Fixed order α_μ^2: large dipoles cutoff needed only in principle
- All orders (resummed bubbles): not integrable singularity $\rightsquigarrow$ "freeze" the coupling or put cutoff
  check independence at the end
- Dynamically generated saturation momentum $Q_s^2 \gg \Lambda_{\rm QCD}^2 \rightsquigarrow$ scale "effectively" setting the argument of the running coupling
  will ensure cutoff independence

POMERON INTERCEPT AND IR SENSITIVITY

Pomeron Intercept (1/5)

- Assumptions
  - linear equation
  - simplified evolution kernel
  - particular running
- What is fastest increase of amplitude?

- “Running coupling evolution equation”

$$\frac{\partial T}{\partial Y} = \alpha(\rho)\left[1+\left(\partial_\rho+\frac{1}{2}\right)^2\right]T \quad \text{with} \quad \rho = \ln 1/\boldsymbol{r}^2\Lambda^2$$

  Can choose more general coefficients or form

- Exact general solution for $\alpha = 1/\rho$ in terms of Airy function

$$T(\rho, Y) = \sum_\omega \exp\left(\omega Y - \frac{\rho}{2}\right)\,\mathrm{Ai}\left(\frac{\omega\rho - 1}{\omega^{2/3}}\right)$$

- Cut infrared contribution $r > r_0 > 1/\Lambda \rightsquigarrow$ boundary condition $T(\rho_0) = 0$
- For given boundary, ω related to zeros of Airy function

$$\omega_n = \frac{1}{\rho_0} - \frac{|\xi_n|}{\rho_0^{5/3}} + \cdots = \alpha(\rho_0) - |\xi_n| \, \alpha^{5/3}(\rho_0) + \ldots$$

- Solution becomes

$$T(\rho, Y) = \sum_n \exp\left(\omega_n Y - \frac{\rho - \rho_0}{2}\right) \mathrm{Ai}(-|\xi_n| + \omega_n^{1/3}(\rho - \rho_0))$$

Pomeron Intercept (4/5)

- Rightmost zero of Airy function at $-|\xi_1| = -2.33 \rightsquigarrow$ Largest ω
  $\omega_1 =$ Pomeron intercept
- In QCD: $\omega_{\mathbb{P}} = 4\ln 2\,\bar{\alpha}$
- $n = 1$ solution dominates up $\rho - \rho_0 \lesssim [\alpha(\rho_0)\,Y]^{2/3}$
  $n \neq 1$ not very physical (oscillations)
- Schrodinger equation: attractive linear potential $\rightsquigarrow$
  Solution in perturbative region strongly dependent on cutoff
- Running coupling BFKL not self-consistent

Pomeron Intercept (5/5)

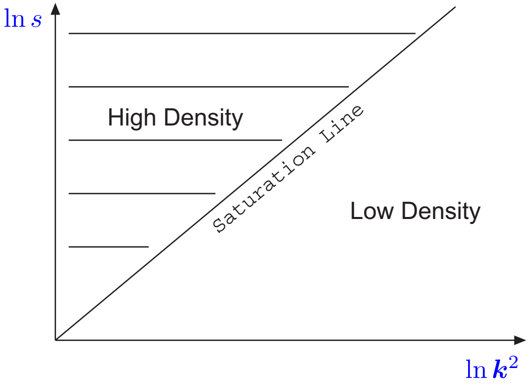
- Assume something milder than “absorptive” boundary
  “Freeze” the coupling

$$\alpha(\rho) = \begin{cases} 1/\beta\rho & \text{for} \quad \rho \gg 1 \\ \mathcal{O}(1) & \text{for} \quad \rho = \rho_0 \sim \mathcal{O}(1) \\ \alpha_0 < 1 & \text{for} \quad \rho = -\infty \end{cases}$$

  with $\alpha(\rho)$ monotonic
- Diffusion to infrared:
  For any given perturbative dipole $\rho \gg \rho_0$, main contribution from region where coupling is strongest: momenta $\sim \Lambda$

THE SATURATION MOMENTUM AND GEOMETRIC SCALING

Logarithmic plane



- Saturation line: transition from low to high density
- $T(\boldsymbol{r}^2 = 1/Q_s^2(Y)) = \text{const}$

Saturation momentum (1/8)

- Enough to analyze linear equations
- Boundary conditions replace non-linear terms

  Caution: b.c are Y-dependent
- Expectation: Non-linear terms $\sim$ cutoff

  Physics around Q_s determined by momenta around Q_s
- Initially assume $\alpha \to \alpha(Q_s)$
  - Leading behavior of saturation momentum
  - All schemes $\to$ same answer

Saturation momentum (2/8)

- Linear running coupling equation

$$\frac{\partial T}{\partial Y} = \frac{1}{\beta \rho_s}\,\chi(1+\partial_\rho)T$$

- Find line $\rho_s(Y)$ along which $T = \mathrm{const}$
  - Change variable $\rho \to z \equiv \rho - \rho_s(Y)$
  - Expand chi function around (yet unknown) γ_s
  - Set derivative of amplitude w.r.t. Y equal to zero
  - Set constant term and coefficient of ∂_z equal to zero

  Two equations determine
  - Anomalous dimension γ_s
  - Saturation momentum $Q_s^2(Y)$

Saturation momentum (3/8)

- Leading Y-dependence of saturation momentum

$$Q_s^2(Y) = \Lambda^2 \exp\left[\sqrt{\frac{2\chi(\gamma_s)}{\beta(1-\gamma_s)}\,(Y+Y_0)}\right] \quad \text{with} \quad \gamma_s = 0.372$$

- $0 < \gamma_s < 1/2$: between DGLAP and Pomeron intercept
- Slower increase: coupling decreases along saturation line
- Consequence of running coupling:

  At high energies the same Q_s for any hadron $\leadsto$

  no $A^{1/3}$ enhancement for a nucleus

- Preasymptotic terms are not negligible
- Expand running coupling to 1st order around Q_s

$$\alpha(\rho) = \frac{1}{\beta\rho_s} - \frac{z}{\beta\rho_s^2} + \ldots$$

Different evolution equation for different schemes

- Can show scheme-independence of first correction

Choose "parent dipole scheme"

Saturation momentum (5/8)

- Solve (approximately) 2nd order P.D.E. with Y-dependent b.c.
- The saturation momentum

$$Q_s^2(Y) = \Lambda^2 \exp\left[\sqrt{\frac{2\chi(\gamma_s)}{\beta(1-\gamma_s)}\,Y} - A\,Y^{1/6}\right]$$

- Scattering amplitude

$$T(z,Y) = Y^{1/6}\exp[-(1-\gamma_s)z]\mathrm{Ai}\left(-|\xi_1| + B\,\frac{z+c}{Y^{1/6}}\right)$$

- Known constants A and B (contain $-|\xi_1|, \chi_s'', \ldots$)

Saturation momentum (6/8)

- Geometric scaling: Within a distance $\sim Y^{1/6}$ (in log-units) amplitude (total cross section) is function only of $z = \ln 1/\boldsymbol{r}^2 Q_s^2$

$$T = \left(\frac{Q_s^2}{Q^2}\right)^{1-\gamma_s} \left(\ln \frac{Q^2}{Q_s^2} + c\right)$$

  Same expression as in fixed coupling case

  Phenomenon appears for momenta higher than Q_s

- Diffusion radius $Y^{1/6}$: much smaller
- Less sensitive to UV: easier to solve numerically
- No way to get geometrical scaling from DGLAP

Saturation momentum (7/8)

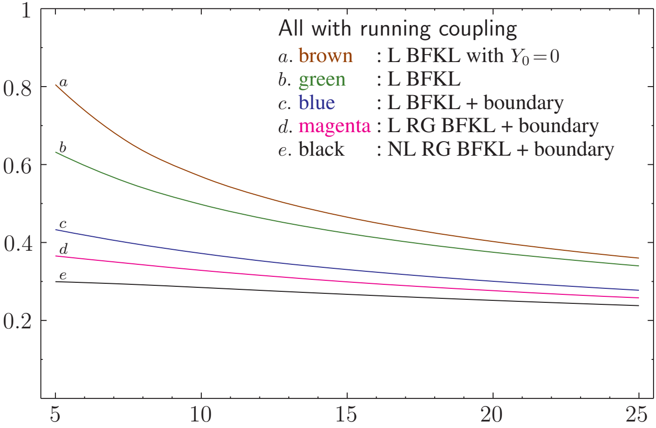
- Full NLO calculation: more terms
- Collinear resummation (DGLAP matching):

$$\gamma(\omega) = \int \mathrm{d}z\, z^{\omega} P_{gg}(z) \Rightarrow \gamma(1) = 0$$

  Topic by itself: see Gavin Salam, hep-ph/9910492
- Cannot really calculate analytically at NLO:

  Coupling along Q_s decreases, NLO converges to running coupling
- Estimate correction for $\lambda_s \equiv \mathrm{d}\ln Q_s^2/\mathrm{d}Y$ to be $\mathcal{O}(\alpha) \sim 30\%$

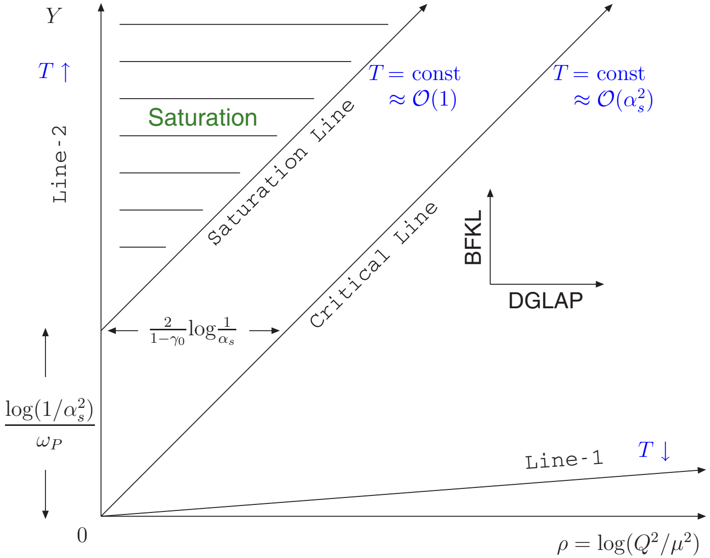
Saturation momentum (8/8)



- More or less what the fits give (GBW,IIM,...): $\lambda_s \simeq 0.3$

RUNNING COUPLING VS. POMERON LOOP EFFECTS

Logarithmic plane



Pathologies

- Extreme sensitivity to ultraviolet: Contribution from momenta $\ln(Q^2/Q_s^2) \lesssim \sqrt{\bar{\alpha} D_s Y} \rightsquigarrow Q^2 \lesssim \dots$
- Reconstruct solution in two steps: violation of unitarity (!)

$$1 \geq T \sim \frac{1}{\alpha^2} T_{\rm a} T_{\rm b} \qquad \text{and for} \quad T_{\rm a} < \alpha^2 \quad \text{then} \quad T_{\rm b} > 1$$

- Absence of Pomeron splittings: Two ladders merge, but how could we have them in the first place?
  - Nucleus target (or even proton?) $\rightsquigarrow$ Many sources $\rightsquigarrow$ Many BFKL pomerons: Initial condition
  - Dynamics: Pomeron splittings $\rightsquigarrow$ Pomeron loops
    Corrections to equations (not present in LO or NLO BK-JIMWLK)

Two-boundary problem (1/2)

- Initially assume fixed coupling
- Solve BFKL with two absorptive boundaries (IR+UV)
- $\Delta = \dfrac{1}{1-\gamma_s} \ln(1/\alpha^2)$ = separation of boundaries
  Within Δ, amplitude drops from $\mathcal{O}(1)$ to $\mathcal{O}(\alpha^2)$
- Look for Y-independent BFKL solution

$$\left[\chi\left(1+\frac{\partial}{\partial z}\right) - \lambda_s \frac{\partial}{\partial z}\right] T = 0$$

- Real combination satisfying boundary conditions (no saddle point)

$$T \sim \exp[-(1-\gamma_{\rm r})\, z] \sin\frac{\pi z}{\Delta}, \qquad \gamma_{\rm i} = \frac{\pi}{\Delta}$$

Two-boundary problem (2/2)

- Real part $\gamma_{\rm r}$ uniquely fixed in terms of $\gamma_{\rm i}$ or Δ or α

$$\lambda_s = \frac{\chi(\gamma)}{1-\gamma} \qquad \text{with} \qquad \mathrm{Im}(\lambda_s) = 0$$

- For large separation of boundaries $\Delta \gg 1 \Leftrightarrow \alpha \ll 1$

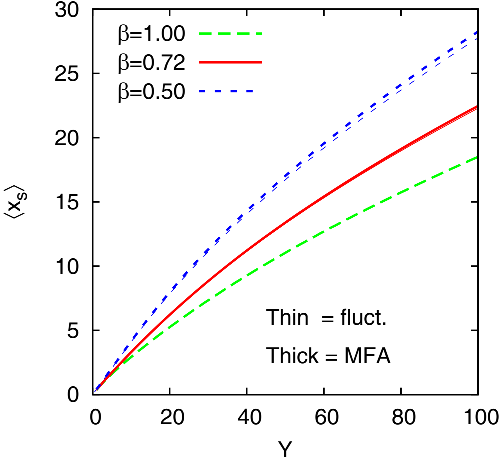
$$\frac{\lambda_s}{\bar{\alpha}} = \frac{\chi(\gamma_s)}{1-\gamma_s} - \frac{\pi^2(1-\gamma_s)\chi''_s}{2\ln^2\alpha^2}$$

- Relative correction is $1/R^2_{\rm eff}$ with $R_{\rm eff}$ = effective transverse space
  True in general
- Running coupling: let $\alpha \to \alpha(Q_s)$

Pomeron loops vs running (1/4)

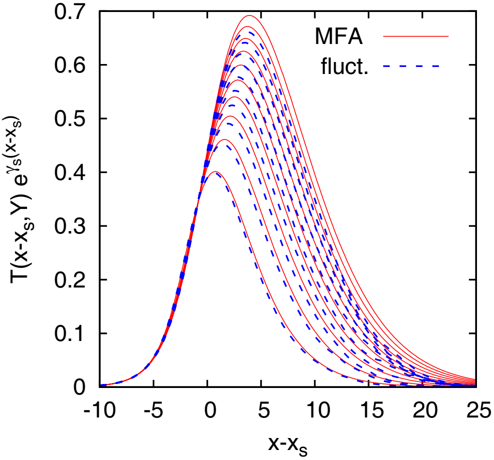
- One of the two effects dominates?
  Or both are important?
  Seek for numerical solutions
- We do not have a theory
  Construct a model on basic principles and include both effects
  Different (but same shape) characteristic function
- Compare pomeron loops + running vs. running

Pomeron loops vs running (2/4)



- No difference in the saturation momentum

Pomeron loops vs running (3/4)



- No difference in the amplitude

Pomeron loops vs running (4/4)

- Up to super high rapidities:
  Pomeron loops + running coupling = running coupling
- Highly non-trivial statement since (for same i.c.)
  Pomeron loops at fixed coupling $\neq$ BK-JIMWLK
- We have used slightly asymmetric initial conditions
  In practice they are: virtual photon - hadron

Pomeron loops vs running: Explanation?

- Compare the two corrections
- Pomeron loops: $\delta\lambda_s \sim 1/\ln^2\alpha \sim 1/R_{\text{eff}}^2$

  $R_{\text{eff}} \sim$ two-boundary width
- Running coupling: $\delta\lambda_s \sim \alpha^{2/3} \sim 1/Y^{1/3} \sim 1/R_{\text{eff}}^2$

  $R_{\text{eff}} \sim$ diffusion radius
- First glance: it seems Pomeron loops are more important
- Diffusion radius grows very slowly with running coupling

  Not really enough “time” to become equal to two-boundary width

  (Contrast to fixed coupling dynamics: diffusion radius $\sim \sqrt{Y}$)